

DIRECT METHODS

KEY IDEA: Discretize OCP $\Rightarrow$ Use NLP solver to find local optimum ⁽⁻⁾

Discretization:

- ① Parametrize state/control trajectories (e.g., with polynomials)
- ② Enforce constraints (dynamics + path inequalities) on a grid ⁽⁺⁾

$$t_0 < t_1 < \dots < t_N$$

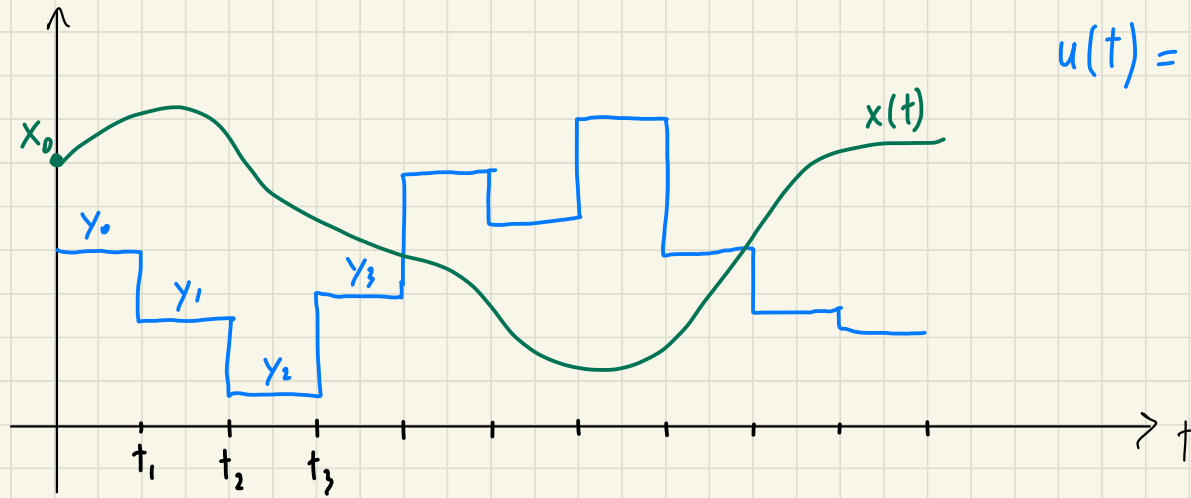
Tons of theory/methods to choose traj. parametrization and time grid so that NLP is i) good approximation of OCP and ii) well conditioned

DIRECT METHOD: FAMILIES

- ① SINGLE SHOOTING (SEQUENTIAL)
 - Discretize only $u(t)$
 - Compute $x(t)$ integrating dynamics
 - No need to have dynamics constraints
- ② COLLOCATION (SIMULTANEOUS)
 - Discretize both $x(t)$ and $u(t)$
 - Enforce dynamics on time grid
- ③ MULTIPLE SHOOTING
 - Discretize all $u(t)$
 - Discretize $x(t)$ only at few points in time
 - Compute intermediate values of x by integration

① SINGLE SHOOTING

Discretize control $u(t)$ on fixed grid $0 = t_0 < t_1 < \dots < t_N = t_f$



$$u(t) = y_i \quad \forall t \in [t_i, t_{i+1}]$$

Compute $x(t)$ from $u(t)$ by integrating dynamics:

$$\dot{x}(t) = f(x, u, t)$$

$$x(0) = x_0$$

SINGLE SHOOTING - NLP

Finite-Dimensional
Nonlinear Program

⇓ Solve with off-the-shelf
solver

minimize
 y

$$\int_0^{t_f} l(x(t; y), u(t; y)) dt + l_f(x(t_f, y))$$

subject to

$$g(x(t_i, y), u(t_i, y), t_i) \leq 0$$

$$i = 0, \dots, N$$

Discretized Path Constraints

Running cost integral typically computed when integrating dynamics:

$$c(t) = \int_0^t l(\cdot, \cdot) dt \quad \Rightarrow \quad \bar{x} = (x, c)$$

$$\begin{cases} \dot{c}(t) = l(\cdot, \cdot) \\ c(0) = 0 \end{cases}$$

$$\dot{\bar{x}} = \begin{bmatrix} f(x, u, t) \\ l(x, u) \end{bmatrix}$$

NUMERICAL INTEGRATION

Given ODE: $\dot{x} = f(x, t)$, $x(0) = x_0$

Compute $x(t) \forall t \in [0, T]$.

If f is Lipschitz continuous $\Rightarrow$ unique solution
 $\approx$ bounded first derivative

Neglect u because typically u is a function of x and/or t .

EXPLICIT EULER

$$\dot{x} = \lim_{h \rightarrow 0} \frac{x(t+h) - x(t)}{h}$$

Take h small

$$h f(x, t) \simeq x(t+h) - x(t) \Rightarrow x(t+h) \simeq x(t) + h f(x, t)$$

OK if h sufficiently small $\Rightarrow$ computationally expensive!
not very accurate! (order 1)

SUMMARY - SINGLE SHOOTING

- Remove x from problem variables
 - Discretize $u(t) = y_i \quad \forall t \in [t_i, t_{i+1}]$
 - Compute x by integrating dynamics $\dot{x} = f(x, u)$
 - $\Rightarrow$ remove dynamics from constraints
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- High-order integration schemes (typically) more efficient (e.g. RK4)
 - Need to differentiate integration scheme to compute gradient of cost function and constraints
 - Implicit schemes typically needed for stiff dynamics

② COLLOCATION

Discretize $x(t)$ with polynomials (e.g., order $k=0$) on **fine** grid =

$$x(t) = s_i$$

$$\forall t \in [t_i, t_{i+1}]$$

One polynomial for each time interval

Replace dynamics $\dot{x} = f(x, u)$ with:

$$\frac{s_{i+1} - s_i}{t_{i+1} - t_i} - f\left(\frac{s_i + s_{i+1}}{2}, y_i\right) = 0 \quad i = 0, \dots, N-1$$

$$c_i(s_i, s_{i+1}, y_i) = 0$$

Approximate cost integral:

$$\int_{t_i}^{t_{i+1}} l(x(t), u(t)) dt \simeq l\left(\frac{s_i + s_{i+1}}{2}, y_i\right) (t_{i+1} - t_i) \triangleq l_i(s_i, s_{i+1}, y_i)$$

COLLOCATION NLP (SPARSE)

$$\underset{s, y}{\text{minimize}} \quad \sum_{i=0}^{N-1} l_i(s_i, s_{i+1}, y_i) + l_f(s_N)$$

subject to

$$s_0 - x_0 = 0$$

initial conditions

$$c_i(s_i, s_{i+1}, y_i) = 0 \quad i = 0 \dots N-1$$

discretized dynamics

$$g_i(s_i, y_i, t_i) \leq 0 \quad i = 0 \dots N$$

discretized path
constraints

Why SPARSE?

③ MULTIPLE SHOOTING

Discretize control $u(t)$ on **coarse** grid $t_0 < t_1 < \dots < t_N$

$$u(t) = y_i \quad \forall t \in [t_i, t_{i+1}]$$

Integrate numerically ODE in each interval $[t_i, t_{i+1}]$ starting from state variable s_i :

$$\dot{x}(t) = f(x_i(t), y_i) \quad t \in [t_i, t_{i+1}]$$

$$x_i(t_i) = s_i$$

Obtain $x_i(t)$. Numerically compute integrals

$$l_i(s_i, y_i) = \int_{t_i}^{t_{i+1}} l(x_i(t), y_i) dt$$

MULTIPLE SHOOTING - NLP

$$\underset{s, y}{\text{minimize}} \quad \sum_{i=0}^{N-1} l_i(s_i, y_i) + l_f(s_N)$$

subject to

$$s_0 - x_0 = 0$$

initial conditions

$$s_{i+1} - \underbrace{x_i(t_{i+1}; s_i, y_i)} = 0$$

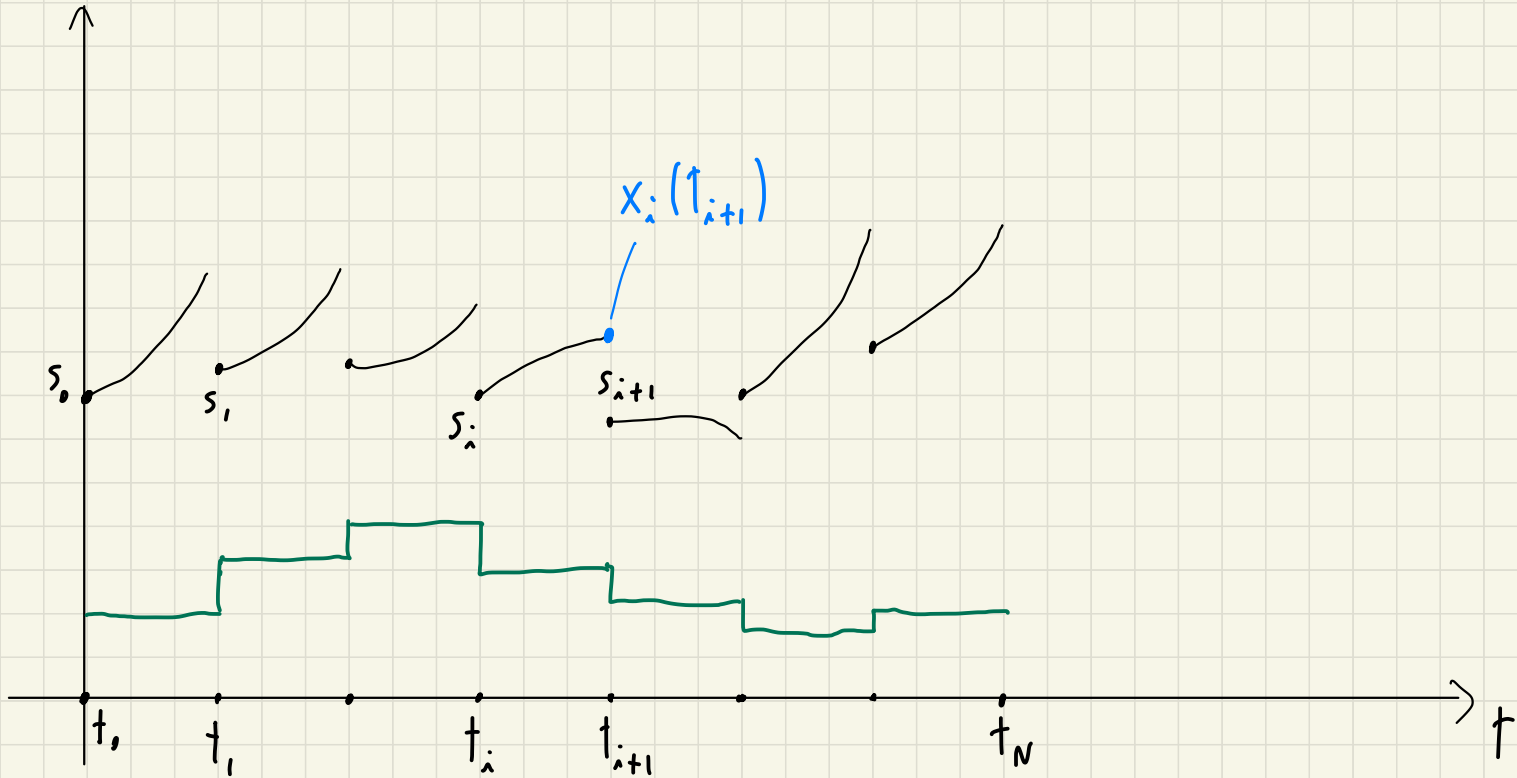
continuity

$$g(s_i, y_i) \leq 0$$

path constraints

Result of numerical
integration from s_i

MULTIPLE SHOOTING SKETCH



DIRECT METHODS - SUMMARY

① SINGLE SHOOTING

- ⊕ Small problem
- ⊕ Need only initial guess for $u(t)$
- ⊕ Can use off-the-shelf ODE solvers
- ⊖ Cannot exploit knowledge of x in initialization
- ⊖ ODE solution can depend very nonlinearly on y
- ⊖ Numerical issues for unstable systems

③ MULTIPLE SHOOTING

- Medium problem
- ⊕ Sparser than ①
- ⊖ Less sparse than ②
- ⊕ Unstable OK
- ⊕ Initialize $x(t)$

② COLLOCATION

- ⊖ Large problem
- ⊖ Cannot adapt time grid
- ⊕ Sparse problem
- ⊕ Work well with unstable systems
- ⊕ Can initialize x